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Relating branes and matrices

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Abstract: We construct a general map between a Dp-brane with magnetic flux and a matrix configuration of D0-branes, by showing how one can rewrite the boundary state of the Dp-brane in terms of its D0-brane constituents. This map gives a simple prescription for constructing the matrices of fuzzy spaces corresponding to branes of arbitrary shape and topology. Since we explicitly identify the D0-brane degrees of freedom on the brane, we also derive the D0-brane charge of the brane in a very direct way including the $\widehat{A}$-genus term. As a check on our formalism, we use our map to derive the abelian-Born-Infeld equations of motion from the action of the D0-brane matrices.

Keywords: M(atrix) Theories, D-branes.

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Contents

1. Introduction

It has been known since the 80's that it is possible to encode the geometry of surfaces embedded in flat space using a collection of matrices [1]. While the original motivation of introducing such a matrix description was to regularize the membrane action, it was found in [2 − 6] that these fuzzy branes appear naturally in string theory as excitations of D0-branes.

In spite of the impressive number of examples of fuzzy geometries formed out of collections of D0-branes [7 − 16], it has never been completely clear what the precise map is between a given Dp-brane configuration and its associated matrix description except in cases with a high degree of symmetry.

In principle, such a map can be derived from the Seiberg-Witten map [17]. This viewpoint has been emphasized in a series of papers by Cornalba, Schiappa and Bordalo [18]. In this approach, one uses the Seiberg-Witten map to replace the algebra of functions on the brane with a star algebra. This star algebra in turn can be replaced by an operator algebra, which can then be interpreted as the algebra of the matrix positions of the D0-branes. The drawback of this method is that defining a star product on a curved brane can be quite complex [19].

What we will show in this paper is that there is actually a more direct method for constructing the D0-brane matrices which correspond to a given Dp-brane background. To derive this map, we take the boundary state of a Dp-brane and manipulate it into the form of the boundary state of a collection of D0-branes. For the case of a flat Dp-brane with uniform flux, an equivalent calculation was performed in [10], where the precise equivalence of the Dp-brane and D0-brane descriptions was demonstrated.

When the Dp-brane is curved, the discussion is somewhat more subtle. To see why, consider the case of a compact Dp-brane with N units of D0-brane charge. In the dual D0-brane description, the positions of the D0-branes are given by $N \times N$ matrices that form a fuzzy version of the original Dp-brane. However, since the matrices are of finite size, there is a mismatch between the infinite number of Dp-brane configurations and the finite number of matrix configurations. This mismatch is most severe when $N = 1$. In this case, the matrices degenerate to scalars representing the location of the single D0-brane and it is not even possible to form a fuzzy geometry. This discrepancy between the Dp-brane picture and the D0-brane picture implies that the exact duality derived in the flat case cannot hold in the general curved case without some caveats.

To understand the origin of this issue, we need to study the properties of string endpoints in the presence of a magnetic field. As has been discussed in [20 − 22], the noncommutative geometry of the D0-branes arises in the Dp-brane picture from the coupling of the string endpoints to the brane geometry. Indeed, the endpoint action is just the ordinary coupling of a charged particle to a background magnetic field. Quantizing this endpoint action gives the usual Landau levels. If the magnetic field is strong, the separation between the levels will be large, and we can assume that the endpoint is constrained to the lowest Landau level. As is well known, constraining a theory to the lowest Landau level naturally leads to a noncommutative geometry [16], which, in this case, we can interpret as the fuzzy geometry of the D0-branes.

Unfortunately, this argument does not explain why the duality shown in [10] works even when the field strength is small. Here the story is more subtle and we give most of the details in section 5. Consider quantizing the world sheet action of a string ending on a flat brane, with uniform magnetic flux. To study the behavior of the endpoint in an unambiguous way, we must impose a cutoff. In section 5, we will consider a lattice regularization in which the endpoint becomes a single lattice site.

It is easy to check in such a model that the endpoint has a non-vanishing expectation value to leave the lowest Landau level. However, all of the physics associated with the excited states of the endpoint lives at the cutoff scale. When the cutoff is removed these excited states decouple from the theory. Hence, even when the background field strength is small, we may assume the endpoint to be constrained to its ground state.

We can now ask whether this same mechanism will work on a curved brane. For a curved brane, we must worry about the potential divergences which arise from the coupling of the endpoint to the curved surface of the brane. These divergences, which also live at the cutoff scale, must be removed by renormalization. In general, the decoupling of the endpoint excitations described above will be destroyed by this renormalization procedure. This will allow the string to see higher Landau levels of the endpoint and, hence, a more complete picture of the brane. Non-trivial Dp-branes with one unit of D0-brane charge must be of this form. We will refer to such branes as smooth branes to distinguish them from fuzzy branes.

On the other hand, for special regulators, the divergences from the curvature of the brane will not interfere with the decoupling of the endpoint excitations. For these branes, the endpoint may be taken to live in the lowest Landau level and much of the original geometry of the brane will be lost. These branes are naturally identified with fuzzy configurations of D0-branes. As we will see, when we perform our boundary state computation, such a regulation can be found. However, we will also argue, that, for a different choice of regulator, the higher modes of the endpoint would not decouple.

We can now make very precise what we mean by relating a brane configuration and matrix configuration. A brane/matrix map simply relates the world sheet theory on a Dp-brane in which the endpoint excitations have decoupled, to a world sheet theory on a matrix configuration of D0-branes. In this sense, the map works for any field strength. Of course if we wish to make sense of the smooth branes, we can only apply the map in cases where the field strength is large or in special cases like the flat Dp-brane with uniform flux.

We now describe the map explicitly. In light of the well known relationship between fuzzy D0-brane configurations and the projection onto to the lowest Landau level, the form of the map should not be surprising. However, projecting an observable onto the lowest Landau level is an ambiguous procedure, and the boundary state computation gives a preferred choice. A different method of projecting onto the lowest Landau level is discussed, for example, in [16]. Our map can be described as follows:

1. Let the world volume Dirac operator in the presence of A_μ be denoted $\mathcal{D}$, and its zero modes given by $\mathcal{D}|\rho\rangle = 0$. Furthermore, pick a basis in which the zero modes are orthonormal.

2. Let the coordinates on the Dp-brane be denoted ξ^α and the embedding into the target space be given by $x^M(\xi)$. Then the coordinate matrices of the D0-branes are given by $\mathbf{X}^M_{\rho_1\rho_2} = \langle\rho_1|x^M(\xi)|\rho_2\rangle$.

Here M is assumed to be a spacial index. Generalizing this construction to time-dependent configurations and configurations in which the dual description is a spacially-dependent matrix is not hard and will be discussed in section 2.3.

This map is, as expected, nothing but a projection of the position operators onto the lowest Landau level. The fact that the lowest energy states are zero modes of the Dirac operator is just a consequence of world sheet supersymmetry. The one subtlety in this map is that some of the zero modes correspond not to D0-branes, but to D$\bar{0}$-branes. As we will show later, the D0-branes correspond to positive chirality modes, whereas, the D$\bar{0}$-branes correspond to negative chirality modes. Note that this correspondence gives an easy derivation of the D0-brane charge formula; the D0-brane charge is simply the index of the Dirac operator. This reproduces the usual formula found in [23 – 26], in the case when the background spacetime is flat. For other methods of reproducing the D0-brane charge from a boundary state computation, see [27, 28].

After deriving this correspondence, we perform a simple check on our results. Starting with the action for a matrix configuration of D0-branes, we use our map to reproduce the usual abelian-Born-Infeld equations of motion. This requires explicitly computing the map in the case of a Dp-brane that has small curvature and a slowly varying field strength. The resulting expressions reproduce the usual formulas for this case [10, 17, 18, 29, 30].

The plan of the paper is as follows: in section 2, we begin by deriving the brane/matrix map for the simpler case of the bosonic string. In section 3, we extend the computation to the superstring. In section 4, we use our map to compute the abelian-Born-Infeld equations of motion from the action of the D0-brane matrices. In section 5, we discuss the decoupling of of the higher modes of the string endpoint. We conclude, in section 6, with some possible future directions.

2. Boundary state analysis

In this section, we derive our brane/matrix map using a manipulation of the boundary state of a Dp-brane. For simplicity, we start with the simpler bosonic argument. We generalize to the superstring in the next section. Many of the techniques used are similar to those in [10, 31, 28].

2.1 Bosonic boundary state basics

We begin with a few basic facts about bosonic boundary states for flat branes. If we imagine a string being emitted from a brane, the boundary of the string world-sheet is simply a loop, $x^M(\sigma)$, living on the brane's surface, where σ runs from 0 to 2π. A basic prescription for computing the boundary state [32] is to sum over the string states representing every such loop, giving each state a weight e^{-S_B}, where S_B is the boundary action of the loop.

Consider a flat brane with Neumann directions, i, and Dirichlet directions, a. Let $X^M(\sigma)$ be the world sheet field, $X^M(z,\bar{z})$, evaluated at $e^{i\sigma}$. The matter part of the Dp-brane boundary state is given, up to normalization, by [32]

$$|\mathcal{B}\rangle = \int \mathcal{D}x^i(\sigma) e^{-S_B(x)} \, |x^i(\sigma); X^a(\sigma) = 0\rangle \,, \tag{2.1}$$

where the state, $|x^i(\sigma); X^a = 0\rangle$, is an eigenvector of $X^M(\sigma)$ with eigenvalues

$$X^i(\sigma)|x^i(\sigma); X^a = 0\rangle \;=\; x^i(\sigma)|x^i(\sigma); X^a = 0\rangle \,, \tag{2.2}$$

$$X^a(\sigma)|x^i(\sigma); X^a = 0\rangle \;=\; 0 \,, \tag{2.3}$$

and the boundary action is given by

$$S_B(x) = \int d\sigma \, A_i(x) \partial_\sigma x^i \,.$$

We now suppose that our Dp-brane is given by some submanifold, Q, embedded in flat space. Let the coordinates on Q be given by ξ^α and let $x^M(\xi^\alpha)$ be the embedding of Q into the background spacetime. We take the M index to run only over spacial indices and the configuration is assumed to be static. We will generalize to dynamical branes later. As with the flat case, there are standard formulas [28, 31−33] for the boundary state of such a brane. We use a slightly different notation, however, which aids our analysis. We have

$$|\mathcal{B}\rangle = \int \mathcal{D}\xi(\sigma)\, e^{-S_B(\xi)} |x^M(\xi(\sigma))\rangle\,, \tag{2.4}$$

where the state, $|x^M(\xi(\sigma))\rangle$, satisfies

$$X^M(\sigma)|x^M(\xi(\sigma))\rangle = x^M(\xi(\sigma))|x^M(\xi(\sigma))\rangle\,, \tag{2.5}$$

and the boundary action is now given by

$$S_B = \int A_\alpha(\xi)\partial_\sigma \xi^\alpha\,.$$

To proceed, rewrite the state, $|x^M(\xi(\sigma))\rangle$, in momentum space

$$|x^M(\xi(\sigma))\rangle = \int \mathcal{D}\pi_M(\sigma)\, \exp\left(i\int d\sigma\, x^M(\xi)\pi_M\right)|\pi_M(\sigma)\rangle\,, \tag{2.6}$$

where $|\pi_M(\sigma)\rangle$ is an eigenvector of the conjugate momenta, $\Pi^M(\sigma) = -i\delta/\delta X^M(\sigma)$, with eigenvalues

$$\Pi_M(\sigma)|\pi_M(\sigma)\rangle = \pi_M(\sigma)|\pi_M(\sigma)\rangle\,.$$

Substituting equation (2.6) into equation (2.4), we can rewrite the boundary state as

$$|\mathcal{B}\rangle = \int \mathcal{D}\xi(\sigma)\mathcal{D}\pi(\sigma) e^{-S_B + i\int d\sigma\, x^M(\xi)\pi_M(\sigma)}|\pi_M(\sigma)\rangle\,. \tag{2.7}$$

The next step in our analysis is to rewrite the path integral over the field, ξ^α, in a more illuminating way. Unfortunately, as written, this path integral is not well defined as there is no kinetic term for ξ^α. This problem is a result of our switch from an X^M eigenbasis to a Π_M eigenbasis. As a simple example of this phenomenon, consider the integral

$$\int_{-\infty}^{\infty} d\pi dx\, e^{ix\pi} e^{-\pi^2/2}\,.$$

Attempting to perform the x integral first is ambiguous. However, performing the π integral gives

$$\int_{-\infty}^{\infty} dx\, \sqrt{2\pi}\, e^{-x^2/2}\,,$$

which is perfectly well defined.

Similarly, in our boundary state computation, if we performed the $\pi_M(\sigma)$ integral first, we would just get back to the X^M basis $|x^M(\sigma)\rangle$. Examining the explicit oscillator representation of $|x^M(\sigma)\rangle$, given in [32], one finds a leading term $\sim e^{-x^2}$, which makes the ξ^α integral well behaved.

To get around this problem, we regularize the ξ^α path integral by adding a small kinetic term to our boundary action

$$S_B = \int d\sigma \left(\frac{1}{2} (\epsilon)\, \partial_\sigma \xi^\alpha \partial_\sigma \xi^\beta g_{\alpha\beta} + A_\alpha(\xi) \partial_\sigma \xi^\alpha \right) . \tag{2.8}$$

This simply gives a small mass, ϵ, to our string endpoint which we will limit to zero at the end of the computation. Note that ϵ need not depend on the regulator we use to make boundary state finite. For example equation (2.7) can me made finite by imposing a cutoff on the π path integral. From this point of view, we have merely added something to our integral to make it better defined and we will limit it away at the end of the calculation. Equivalently, one can think of this small kinetic term as defining the measure for the ξ integral in a natural, covariant way.

On the other hand, there is a very real sense in which ϵ could be thought of as depending on the cutoff. It is easy to construct a short distance cutoff in which the endpoint of the string obtains a mass proportional to the cutoff. In such a scenario, the limit as ϵ and the cutoff go to zero will produce many new terms in the boundary state which depend on excited states of the endpoint.

We will not consider this possibility here, although it is interesting, and, if done properly, might give some insight into the boundary state of smooth branes. For now, however, we will consider ϵ a parameter which is limited to zero at fixed cutoff.

2.2 Interpreting in terms of D0-branes

To express the boundary state in terms of D0-brane degrees of freedom, we rewrite the path integral over ξ^α in terms of a trace over the hamiltonian. Putting

$$\mathcal{L} = \frac{1}{2} (\epsilon)\, \partial_\sigma \xi^\alpha \partial_\sigma \xi^\beta g_{\alpha\beta} + A_\alpha(\xi) \partial_\sigma \xi^\alpha + i x^M(\xi) \pi_M(\sigma), \tag{2.9}$$

the hamiltonian is given by

$$\mathcal{H} = \frac{1}{2\epsilon} (P^\alpha - A^\alpha) g_{\alpha\beta} (P^\beta - A^\beta) - i x^M(\xi) \pi_M(\sigma) . \tag{2.10}$$

We can then use the standard identity,

$$\int \mathcal{D}\,\xi e^{-S} = \mathrm{TrP} e^{-\int d\sigma \mathcal{H}(\sigma)} , \tag{2.11}$$

where P denotes path ordering and the trace, Tr, is over the the space of states. It is convenient to define

$$\mathcal{H}_0 = \frac{1}{2} (P^\alpha - A^\alpha) g_{\alpha\beta} (P^\beta - A^\beta) ,$$

so that $\mathcal{H} = \mathcal{H}_0/\epsilon - i x^M(\xi) \pi_M$. Also, it is useful to pick a basis in which $\mathcal{H}_0$ is diagonal,

$$\mathcal{H}_0 |\lambda\rangle = E_\lambda |\lambda\rangle .$$

Note that, at the quantum level, there is an ordering ambiguity in how one defines the hamiltonian. For the superstring, this will be less of a problem, as the hamiltonian is naturally written as the square of the Dirac operator. Even in the bosonic case, though, covariance puts some constraints on the operator form of $\mathcal{H}_0$.

Using equation (2.11), we can rewrite our path integral as

$$\mathrm{Tr}\,\mathrm{P}\exp\left\{-\int d\sigma\left(\frac{\mathcal{H}_0}{\epsilon}-ix^M\pi_M\right)\right\}.$$

Consider expanding out the exponent in powers of $x^M\pi_M$. For illustration purposes, we do this for the second order term. The manipulations for the other terms are similar. Taking only two powers of $x^M\pi_M$, we get the expression

$$\sum_\lambda\int_0^{2\pi}d\sigma_1 d\sigma_2\ \theta(\sigma_2-\sigma_1)\langle\lambda|e^{-(2\pi-\sigma_2)\mathcal{H}_0/\epsilon}\left(ix^{M_2}\pi_{M_2}(\sigma_2)\right)e^{-(\sigma_2-\sigma_1)\mathcal{H}_0/\epsilon}\times \\ \times\left(ix^{M_1}\pi_{M_1}(\sigma_1)\right)e^{-\sigma_1\mathcal{H}_0/\epsilon}|\lambda\rangle\,, \tag{2.12}$$

where $\theta(\sigma_2-\sigma_1)$ is a step function which enforces the path ordering. Consider the effect of the various insertions of

$$e^{-\delta\sigma\mathcal{H}_0/\epsilon}=\sum_\lambda|\lambda\rangle e^{-\delta\sigma E_\lambda/\epsilon}\langle\lambda|\,.$$

Note that as $\epsilon\to 0$, the term $e^{-\delta\sigma E_\lambda/\epsilon}$ vanishes unless $E_\lambda=0$. Hence in the small ϵ limit, we can write

$$\lim_{\epsilon\to 0}e^{-\delta\sigma\mathcal{H}_0/\epsilon}=\sum_\rho|\rho\rangle\langle\rho|\,, \tag{2.13}$$

where the $|\rho\rangle$ are the states with zero energy. Unfortunately, in most interesting cases there are, in fact, no states of zero energy. However, one can fix this problem by writing

$$\mathcal{H}_0=\mathcal{H}_0'+\mathcal{E}_0\,,$$

where $\mathcal{E}_0$ is the energy of lowest state and $\mathcal{H}'$ now has an eigenstate with zero energy. One can then perform the calculation as before, using $\mathcal{H}'$ instead of $\mathcal{H}$, since the constant term only affects the overall normalization of the boundary state. As we are not dealing with the issue of finding the correct normalization of the boundary state, we will ignore this point and simply assume at least one state with zero energy. When we work with the superstring, we will see that there are *typically* states with zero energy.

Another potential problem arises if there is a continuum of energies instead of a gap above the ground state energy. In this case, the limit $\epsilon\to 0$ is not well defined and our analysis is not legitimate. This occurs, for example, in the case of an infinite flat D2-brane with no flux, where there is no D0-brane interpretation.

Using the formula (2.13) in our expression (2.12), we get

$$\sum_{\rho_1\rho_2}\int_0^{2\pi}d\sigma_1 d\sigma_2\theta(\sigma_2-\sigma_1)\langle\rho_1|ix^{M_2}\pi_{M_2}(\sigma_2)|\rho_2\rangle\langle\rho_2|ix^{M_1}\pi_{M_1}(\sigma_1)|\rho_1\rangle\,, \tag{2.14}$$

where, as before, the $|\rho_i\rangle$ are assumed to have zero energy. To put this expression in a simpler form, define the matrices

$$\mathbf{X}^M_{\rho_1\rho_2} = \langle\rho_1|x^M|\rho_2\rangle\,. \tag{2.15}$$

Equation (2.14) can then be written as

$$(i)^2\int_0^{2\pi} d\sigma_1 d\sigma_2\;\theta(\sigma_2-\sigma_1)\mathrm{tr}\left[\mathbf{X}^{M_2}\mathbf{X}^{M_1}\right]\pi_{M_2}(\sigma_2)\pi_{M_1}(\sigma_1)|\lambda_1\rangle\,, \tag{2.16}$$

which is just the second order term in

$$\mathrm{trP}\exp\left\{i\int d\sigma\,\mathbf{X}^M\pi_M(\sigma)\right\}.$$

The other terms in the expansion follow from a similar argument. The full boundary state is given by

$$\int\mathcal{D}\pi(\sigma)\;\mathrm{trP}\exp\left\{i\int d\sigma\,\mathbf{X}^M\pi_M(\sigma)\right\}|\pi^M\rangle\,. \tag{2.17}$$

Switching back to the X^M basis gives our final expression

$$|\mathcal{B}\rangle = \mathrm{trP}\exp\left\{i\int d\sigma\,\mathbf{X}^M\Pi_M(\sigma)\right\}|x^M=0\rangle\,.$$

This expression is the usual formula for the boundary state of a collection of bosonic D0-branes with matrix-valued positions $\mathbf{X}^M$ [28, 31–33]. Note that we now have an explicit map between the brane configuration and the matrix-configuration of the D0-branes given by eq. (2.15). We now present two familiar examples to illustrate the map.

Example 1: constant F_{ij} on a flat D2-brane. Consider, as an example, a flat D2-brane with constant $F_{12}=f$. For this case, the equivalence of the D2-brane boundary state and the D0-brane boundary state was shown in [10]. We can take the gauge field to be $A_1=-fX^2$ with $A_2=0$. Note that the hamiltonian, $\mathcal{H}_0$ is simply the hamiltonian for a spinless particle in a constant magnetic field. This is just the ordinary Landau problem. The states in the lowest energy level are labeled by $|k\rangle$. The matrices $\mathbf{X}^1$ and $\mathbf{X}^2$ are given by

$$\mathbf{X}^1_{kk'} = -i\frac{\partial}{\partial k}\delta(k-k')\,, \tag{2.18}$$

$$\mathbf{X}^2_{kk'} = \frac{k}{F}\delta(k-k')\,. \tag{2.19}$$

Hence $[\mathbf{X}^2,\mathbf{X}^1]=\frac{i}{f}\mathbf{1}$, where $\mathbf{1}$ is the identity matrix. This is the standard configuration of D0-branes said to correspond to a D2-brane.

Example 2: spherical D2-brane with uniform flux. Now consider a spherical D2-brane with constant flux. For this setup, $\mathcal{H}_0$ is given by the hamiltonian for a spinless particle constrained to a sphere with a constant magnetic field pointing radially out from the sphere. This problem is quite old, and was solved, for example, in [34]. A more elegant solution is presented in [35] and a modern discussion is given in [22]. For our purposes we need only know that the hamiltonian takes the form

$$\mathcal{H}_0 = \frac{2\pi\alpha'}{2R^2}\left(L^2 - \frac{N^2}{4}\right),$$

where the L^i are the generators of angular momentum in the presence of the gauge field and $i = 1, 2, 3$. The ground state is an $N+1$ dimensional representation of the Ls and we can compute the matrices $\mathbf{X}^i$ using the Wigner Eckart theorem. Let the ground states be denoted $|\ell = N/2, m\rangle$, with $-N/2 \leq m \leq N/2$. We have

$$\mathbf{X}^i_{m_1 m_2} = \langle \ell, m_1|x^i|\ell, m_2\rangle = \frac{\langle \ell, m_1|\sum_i L^i x^i|\ell, m_1\rangle}{\ell(\ell+1)}\langle \ell m_1|L^i|\ell m_2\rangle\,. \tag{2.20}$$

Note the matrix element $\langle \ell, m_1|\sum_i L^i x^i|\ell, m_1\rangle$ is independent of m_1. An explicit computation gives

$$\langle \ell, m_1|\sum_i L^i x^i|\ell, m_1\rangle = R\,\frac{N}{2}\,.$$

Since $\langle \ell m_1|L^i|\ell m_2\rangle$ is just the matrix representation of L^i, we can write

$$\mathbf{X}^i = \frac{R}{(N/2+1)}\,\mathbf{J}^i,$$

where the $\mathbf{J}^i$ are the generators of the $N+1$ dimensional representation of SU(2). This is the standard representation of a sphere using D0-branes [3, 11].

Note that the dimension of the representation is $N+1$, implying that a Dp-brane with N units of flux is built from $N+1$ D0-branes in the bosonic theory. This disagrees with the expected result from the superstring, but, since there is no charge associated with the D0-brane in the bosonic theory, it is not inconsistent. We will see that, in the superstring analysis, we always reproduce the correct D0-brane charge.

2.3 Time dependent backgrounds

The only new ingredient in a time dependent background is that the hamiltonian, $\mathcal{H}_0$, now depends on X^0. This implies that we must make the replacement

$$e^{-\delta\sigma\mathcal{H}_0/\epsilon} \to P\exp\left\{-\int_{\sigma_1}^{\sigma_2} d\sigma\;\frac{\mathcal{H}_0\left(X^0(\sigma)\right)}{\epsilon}\right\}.$$

For a general time dependent hamiltonian, we cannot evaluate this object. However, if we assume that we can follow the eigenvectors of the hamiltonian in time and that no eigenvectors appear or disappear, we can rewrite this expression as follows: Label the eigenvectors in the ground state at time X^0 by $|\rho(X^0)\rangle$. Then using the adiabatic theorem we get

$$P\exp\left\{-\int_{\sigma_1}^{\sigma_2} d\sigma\;\frac{\mathcal{H}_0(X^0(\sigma))}{\epsilon}\right\} = \sum_\rho |\rho(X^0(\sigma_2))\rangle\langle\rho(X^0(\sigma_1))|. \tag{2.21}$$

The difference between this formula and the time independent formula (2.13), can be accounted for by modifying the definition of the matrices,

$$\mathbf{X}^M_{\rho_2\rho_1}(X^0) = \langle \rho_2(X^0)|x^M|\rho_1(X^0)\rangle\,,$$

so that the matrices now depend on time. We could also consider cases where $\mathcal{H}_0$ depends on some spacial direction. The same argument as above will then work, giving spacially dependent matrices.

3. Superstring boundary states

In this section, we generalize the results of the previous section to the superstring. We begin, as before, with a review of the form of the boundary state and then show how to rewrite it in terms of D0-branes. Much of the discussion is parallel with the bosonic case, however, we find a new contribution when two operators collide in the path integral.

3.1 The SUSY boundary state for flat branes

To include the fermion fields, we define boundary fields, $\Psi^M(\sigma)$ and $\widetilde{\Psi}^M(\sigma)$, to be the fermion fields, $\Psi(z)$ and $\widetilde{\Psi}(\bar{z})$, at $z = e^{i\sigma}$ and $\bar{z} = e^{-i\sigma}$ respectively. We then define the field, $\Psi^M_\pm = \Psi^M \pm i\widetilde{\Psi}^M$. Before defining the complete boundary state, it is standard to define [32]

$$|\mathcal{B};\pm\rangle = \int \mathcal{D}x^i \mathcal{D}\psi^i e^{-S_B(x,\psi)} |x^i; X^a = 0\rangle |\psi^i; \psi^a = 0, \pm\rangle\,,$$

where the state $|\psi^i; \psi^a = 0; \pm\rangle$, which is a function of the classical Grassmann field $\psi^i(\sigma)$, has the properties

$$\Psi^i_\pm(\sigma)|\psi^i; \psi^a = 0; \pm\rangle = \psi^i(\sigma)|\psi^i; \psi^a = 0; \pm\rangle\,, \tag{3.1}$$

$$\Psi^a_\pm(\sigma)|\psi^i; \psi^a = 0; \pm\rangle = 0\,. \tag{3.2}$$

The boundary action, S_B, is given by

$$S_B = \int d\sigma \left(A_i \partial_\sigma x^i - i\frac{1}{2} F_{ij}\psi^i\psi^j \right).$$

The complete boundary state is then given by [32]

$$|\mathcal{B}\rangle = P\widetilde{P}_+|\mathcal{B};+\rangle_{\mathrm{NS}} + P\widetilde{P}_-|\mathcal{B};+\rangle_{\mathrm{RR}}\,,$$

where P and $\tilde{P}$ are GSO projectors. Since we will not be concerned with the GSO projections, we will simply work with $|\mathcal{B};+\rangle$. For convenience, we will also drop the $\pm$ labels as they will not play any roll in the discussion.

3.2 The boundary state for curved branes

Generalizing to curved branes is straightforward. For a nice discussion, see [28, 31, 33]. As before we take the coordinates on the world volume of the brane to be ξ^α and we take the spinors, ψ^i, to live on the world volume, with i a tangent space index. We take, as before, $x^M(\xi)$ to be the embedding of the brane into spacetime, and $g_{\alpha\beta}(\xi)$ to be the induced metric on the brane. Finally, we introduce a vielbein $e^\alpha{}_i(\xi)$. We can then define the boundary state as

$$\int \mathcal{D}\xi^\alpha\, \mathcal{D}\psi^\alpha\, e^{-S_B} |x^M(\xi)\rangle |\Psi^M = \frac{\partial x^M}{\partial \xi^\alpha} e^\alpha_i(\xi)\psi^i\rangle\,,$$

where

$$S_B = \int d\sigma \left(A_\alpha(\xi)\partial_\sigma \xi^\alpha + i\frac{1}{2} F_{\alpha\beta} e^\alpha{}_i e^\alpha{}_j \psi^i \psi^j \right).$$

It is useful to rewrite the state, $|\Psi^M = \frac{\partial x^M}{\partial \xi^\alpha} e_i^\alpha(\xi)\psi^i\rangle$, in Fourier space:

$$|\Psi^M = \frac{\partial x^M}{\partial \xi^\alpha} e_i^\alpha(\xi)\psi^i\rangle = \int \mathcal{D}\chi^M e^{-\int d\sigma \, \frac{\partial x^M}{\partial \xi^\alpha} e_i^\alpha(\xi)\psi^i \chi_M} |\chi_M\rangle \,, \tag{3.3}$$

where the state, $|\chi_M\rangle$, satisfies

$$\Psi^M(\sigma)|\chi_M\rangle = \frac{\delta}{\delta \chi^M(\sigma)} |\chi_M\rangle \,.$$

This allows us to write

$$|\mathcal{B}\rangle = \int \mathcal{D}\xi^\alpha \, \mathcal{D}\psi^\alpha \, \mathcal{D}\chi^M \, \mathcal{D}\pi^M \, e^{-S_B + \int d\sigma \quad i x^M \pi_M - \psi^i e^\alpha{}_i \frac{\partial x^M}{\partial \xi^\alpha} \chi_M} \; |\pi^M\rangle |\chi^M\rangle \,. \tag{3.4}$$

We would now like to perform the path integral over the ξ and ψ fields, but, as in the bosonic case, the path integral is not well-defined. As before, we fix this problem by adding a small kinetic term for ξ^α and ψ^i. Our exponent becomes $-\int d\sigma \mathcal{L}$, where $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L} = {} & \frac{1}{2}\,(\epsilon)\,\partial_\sigma \xi^\alpha \partial_\sigma \xi\beta g_{\alpha\beta} + i\frac{1}{2}\,(\epsilon)\,\psi^i(\delta_{ij}\partial_\sigma - \partial_\sigma \xi^\alpha \omega_{\alpha ij})\psi^j + \\ & + A_\alpha \partial_\sigma \xi^\alpha + i\frac{1}{2} F_{\alpha\beta}\psi^\alpha \psi^\beta - i x^M \pi_M + \psi^i e^\alpha{}_i \frac{\partial x^M}{\partial \xi^\alpha} \chi_M \,. \end{aligned} \tag{3.5}$$

Note that we have introduced the spin connection, ω_α^{ij}, in order to make a covariant kinetic term for ψ^i. This action is simply the supersymmetric extension of the original action. We now rewrite the path integral over ξ and ψ in the hamiltonian formalism as we did in the bosonic case.

3.3 Interpreting the boundary state in terms of D0-branes

As we did in the bosonic case, we rewrite our path integral as

$$\mathrm{PTr} e^{-\int d\sigma \mathcal{H}} \,, \tag{3.6}$$

where the hamiltonian, $\mathcal{H}$, corresponding to the lagrangian (3.5) is given by

$$\mathcal{H} = \frac{1}{2\epsilon}\mathcal{D}^2 - i x^M \pi_M + \frac{1}{\sqrt{2\epsilon}} \Gamma^i e^\alpha{}_i \frac{\partial x^M}{\partial \xi^\alpha} \chi_M \,.$$

Here $\mathcal{D}$ is the ordinary Dirac operator,

$$\mathcal{D} = \Gamma^I(-i\partial_I - A_I) \,,$$

and the Γ^I satisfy $\{\Gamma^I, \Gamma^J\} = 2\delta^{IJ}$ as usual. The factor of $1/\sqrt{2\epsilon}$ comes from our non-standard normalization of the fermion kinetic term. As in the bosonic case, define

$$\mathcal{H}_0 = \frac{1}{2}\mathcal{D}^2 \,.$$

Let the states, $|\lambda\rangle$, be the eigenstates of $\mathcal{H}_0$, so that $\mathcal{H}_0|\lambda\rangle = E_\lambda|\lambda\rangle$. As we found in the bosonic case, the hamiltonian will project onto the states of zero energy. As before, denote the states of zero energy by $|\rho\rangle$. Since $\mathcal{H}_0$ is just the square of the Dirac operator, we have

$$\mathcal{D}|\rho\rangle = 0\,.$$

For the term, $ix^M\pi_M$, the analysis proceeds in the same way as for the bosonic case. The $1/\sqrt{2\epsilon}$ term, however, needs to be treated in a slightly different way. It is convenient to note that

$$\Gamma^i e^\alpha{}_i \frac{\partial x^M}{\partial \xi^\alpha}\chi_M = [\mathcal{D}, x^M(\xi)]\chi_M\,,$$

which implies that this term vanishes between states of zero energy,

$$\langle\rho|\frac{1}{\sqrt{2\epsilon}}[\mathcal{D}, x^M(\xi)]\chi_M|\rho'\rangle = 0\,,$$

as the Dirac operator kills either the state on the left or on the right. Since, as in the bosonic case, one will have a factors of $e^{-\delta\sigma\mathcal{H}_0/\epsilon}$ around each insertion of $1/\sqrt{2\epsilon}\,[\mathcal{D}, x^M(\xi)]\chi_M$, and these factors project onto the ground state, one might suppose that this term will not contribute. Indeed, when the term is separated from other such insertions, this is true. However, we get a nonzero contribution when two $1/\sqrt{2\epsilon}\,[\mathcal{D}, x^M(\xi)]\chi_M$ terms collide.

Consider a term in the expansion of the exponential (3.6), with two neighboring insertions of $1/\sqrt{2\epsilon}\,[\mathcal{D}, x^M(\xi)]\chi_M$, which are separated by $\Delta\sigma = \sigma_2 - \sigma_1$

$$\frac{1}{\sqrt{2\epsilon}}[\mathcal{D}, x^{M_1}]\chi_{M_1}(\sigma_2)\left(e^{-\Delta\sigma\mathcal{H}_0/\epsilon}\right)\frac{1}{\sqrt{2\epsilon}}[\mathcal{D}, x^{M_2}]\,\chi_{M_2}(\sigma_1)\,. \tag{3.7}$$

If we suppose that these insertions are not near any other insertions, we may assume that there are projectors onto the states of zero energy to the left and right. This allows us to rewrite our expression as

$$\begin{aligned} x^{M_1}\chi_{M_1}(\sigma_2)\frac{1}{2\epsilon}\mathcal{D}^2\left(e^{-\Delta\sigma\mathcal{H}_0/\epsilon}\right)x^{M_2}\,\chi_{M_2}(\sigma_1) &= \\ = -\,x^{M_1}\chi_{M_1}(\sigma_2)\mathcal{H}_0\sum_\lambda|\lambda\rangle\frac{e^{-\Delta\sigma E_\lambda/\epsilon}}{\epsilon}\langle\lambda|\,x^{M_2}\,\chi_{M_2}(\sigma_1)\,. \end{aligned} \tag{3.8}$$

Note, however, that

$$\lim_{\epsilon\to 0}\frac{e^{-\Delta\sigma E_\lambda/\epsilon}}{\epsilon} = \frac{1}{E_\lambda}\delta(\Delta\sigma),$$

unless $E_\lambda = 0$. Fortunately, the $E_\lambda = 0$ states do not contribute, as they are killed by the extra factor of $\mathcal{H}_0$ in equation (3.8).

Hence, integrating (3.8) over $\Delta\sigma$ gives

$$\begin{aligned} -\,x^{M_1}\chi_{M_1}\sum_{\lambda\neq 0}|\lambda\rangle\langle\lambda|\,x^{M_2}\,\chi_{M_2} &= -\,x^{M_1}\chi_{M_1}\left(1 - \sum_\rho|\rho\rangle\langle\rho|\right)x^{M_2}\,\chi_{M_2}(\sigma_1) \\ &= x^{M_1}\chi_{M_2}\sum_\rho|\rho\rangle\langle\rho|\,x^{M_2}\,\chi_{M_1}\,, \end{aligned} \tag{3.9}$$

where in the last line we have used the antisymmetry of the two χ_M fields. Consider that the term (3.9) will always appear sandwiched between projectors $\sum_\rho |\rho\rangle\langle\rho|$. Taking a bra from the left projector and a ket from the right projector, we get the matrix element

$$\langle\rho_2|x^{M_1}\chi_{M_2}\sum_\rho |\rho\rangle\langle\rho|\, x^{M_2}\,\chi_{M_1}|\rho_1\rangle = \mathbf{X}^{M_1}_{\rho_2\rho}\mathbf{X}^{M_1}_{\rho\rho_1}\chi_{M_2}\chi_{M_1} = \frac{1}{2}\left[\mathbf{X}^{M_2},\mathbf{X}^{M_1}\right]_{\rho_2\rho_1}\chi_{M_2}\chi_{M_1}\,. \tag{3.10}$$

The upshot of this computation is that, including the contributions from the $ix^M\pi_M$ terms, our boundary state becomes

$$\int \mathcal{D}\pi\,\mathcal{D}\chi\,\mathrm{trP}\exp\Big\{i\,\mathbf{X}^M\pi_M + \frac{1}{2}[\mathbf{X}^M,\mathbf{X}^N]\chi_M\chi_N\Big\}|\pi^M\rangle|\chi^M\rangle\,. \tag{3.11}$$

This is precisely the boundary state of a collection of D0-branes [28, 31] up to one subtlety which we now describe.

Since the states $|\rho\rangle$ satisfy $\mathcal{D}|\rho\rangle = 0$, they can be organized into positive and negative chirality states. Since none of the objects we are considering switch the chirality of the states, the boundary state is actually a sum of two terms, one that uses only positive chirality states and one that uses only negative chirality states.

Up till now, we have been working in the NS sector. However, if we consider the same calculation in the RR sector, we need to account for the change in boundary conditions of the fermions. This is accomplished by putting a factor of the Γ inside the over trace. This has the effect of switching the sign of the negative chirality states and, hence, their RR charge. The interpretation is simple: the positive chirality states represent D0-branes while the negative chirality states represent D$\bar{0}$-branes. Note, that this implies that the D0-brane charge of the brane is given by the index theorem [36, 37]

$$\#\mathrm{D0s} - \#\mathrm{D\bar{0}}s = \mathrm{index}(\mathcal{D}) = \int \widehat{A}\wedge \mathrm{ch}(F)\,,$$

where $\widehat{A}$ is the A-roof genus of the brane manifold. This correctly reproduces the D0-brane charge formula found in [23 – 26], in the case when the spacetime background is flat.

To summarize, we have found a simple map between a brane configuration and a matrix configuration of D0 and D$\bar{0}$-branes. One simply finds the zero modes of the Dirac operator, which we denote $|\rho\rangle$, and then computes the matrix

$$\mathbf{X}^M_{\rho_1\rho_2} = \langle\rho_1|x^M(\xi)|\rho_2\rangle\,.$$

4. Derivation of the Born-Infeld action from the matrix action

As a concrete check that our map makes sense, we derive the equations of motion of the abelian-Born-Infeld action using the equations of motion of the matrices. This sort of derivation is similar to those found in [10, 17, 18, 30] and is essentially a rephrasing in matrix language of the fact that the form of the Born Infeld action is invariant under the Seiberg-Witten map [17].

Recall that the the Born-Infeld (BI) action is derived in the approximation of slowly varying field strength [38, 39]. Thus, we can think of the BI action as giving the equations of motion for a small perturbation of the gauge field around a constant F_{MN} background. Hence, to derive the BI action from the matrix action we must calculate the matrices associated to a constant F_{MN} background plus a small fluctuation.

The plan of the calculation is as follows. First, we find the zero modes of the Dirac operator in a constant F_{MN} background and use our map to construct the associated matrices. We then perturb this background by δA_M and calculate the change in the matrices. Finally, we expand the matrix equation of motion to lowest order in δA_M and check that the resulting equation is equivalent to the BI-action equations of motion. We can also consider small perturbations in the location of the brane. These turn out to be much more trivial so we will only sketch how they work out at the end.

4.1 The matrices for a constant F_{MN} background

Our first goal is to solve for the zero modes when F is constant. We will add on fluctuations later. Our hamiltonian is given by

$$H = -\frac{1}{2}(\partial_J - iA_J)^2 + i\frac{1}{4}\Gamma^J\Gamma^K F_{JK}.$$

By using rotations we can assume that the only non-vanishing components of F_{MN} are the $f_i = F_{2i-1\,2i}$. We chose the gauge $A_{2i-1} = -f_i\, x^{2i}$.

Furthermore, we take our states to be eigenvectors of $\Sigma^{2i-1\,2i} = -\frac{i}{4}\Gamma^{2i-1}\Gamma^{2i}$, with eigenvalue $s_i = \pm\frac{1}{2}$. It will also be useful to define $p_i = x_{2i-1}$ and $q_i = x_{2i}$. With these conventions, the eigenstates of H are given by the eigenfunctions of the simple harmonic oscillator in the even directions and plane waves in the odd directions.

$$\begin{aligned} |\Psi_{k,n,s}\rangle = \prod_i \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{f_i}}{2^{n_i}(n_i)!\sqrt{\pi}}\right)^{1/2} e^{ik_ip_i}\mathcal{H}_{n_i}\left(\sqrt{f_i}\left(q_i + \frac{k_i}{f_i}\right)\right) \times \\ \times \exp\left\{-\frac{f_i}{2}\left(q_i+\frac{k_i}{f_i}\right)^2\right\}|s\rangle\,, \end{aligned} \tag{4.1}$$

where the $\mathcal{H}_n$ are the Hermite polynomials. The energies are given by $\mathcal{E}_{k,n,s} = \sum_i f_i(n_i - s_i + \frac{1}{2})$. Note that the k, n and s labels without the i subscript are used as collective indices.

The lowest energy states have zero energy. They are given by taking $n_i = 0$ and $s_i = \frac{1}{2}$ and will be denoted $|\Psi_k\rangle$. These states determine

$$\mathbf{X}_0^{2i-1} = \langle\Psi_k|p_i|\Psi_{k'}\rangle = -i\partial_{k'_i}\delta(k-k')\,, \tag{4.2}$$

$$\mathbf{X}_0^{2i} = \langle\Psi_k|q_i|\Psi_{k'}\rangle = -\frac{k_i}{f_i}\delta(k-k')\,, \tag{4.3}$$

where $\delta(k-k') = \prod_i \delta(k_i - k'_i)$. This determines the commutation relation

$$[\mathbf{X}_0^{2i-1}, \mathbf{X}_0^{2i}] = -\frac{i}{f_i}\mathbf{1}\,.$$

Where $\mathbf{1}$ is the matrix $\delta(k-k')$. This is as we found in the bosonic theory. Covariantly, we can write this as

$$[\mathbf{X}_0^M, \mathbf{X}_0^N] = i\,\theta^{MN}\,, \tag{4.4}$$

where θ^{MN} is the inverse of F_{MN}.

4.2 The shift in the matrices under a shift δA_M

We now want to consider small fluctuations, δA_J, about the constant F background. Computing the change in the matrices under this shift is straightforward, but requires some algebra. The reader interested in just the results can skip to subsection 4.4 after examining equations (4.12) and (4.17). Consider the Dirac operator for our background

$$\mathcal{D} = \Gamma^J(-i\partial_J - A_J)\,.$$

Under our perturbation, we get $\mathcal{D} \to \mathcal{D} + \delta\mathcal{D}$, where

$$\delta\mathcal{D} = -\Gamma^J \delta A_J\,.$$

Our states are also perturbed: $|\Psi_k\rangle \to |\Psi_k\rangle + |\delta\Psi_k\rangle$. Since we want to find the new ground states, we must solve

$$(\mathcal{D} + \delta\mathcal{D})(|\Psi_k\rangle + |\delta\Psi_k\rangle) = 0\,,$$

which to lowest order gives

$$\mathcal{D}|\delta\Psi_k\rangle = -\delta\mathcal{D}|\Psi_k\rangle\,, \tag{4.5}$$

or, equivalently,

$$2H|\delta\Psi_k\rangle = -\mathcal{D}\delta\mathcal{D}|\Psi_k\rangle\,, \tag{4.6}$$

which has the solution

$$|\delta\Psi_k\rangle = \sum_{\mathcal{E}_{k',n,s}\neq 0} |\Psi_{k,n,s}\rangle \frac{1}{2\mathcal{E}_{k',n,s}} \langle\Psi_{k,n,s}|\mathcal{D}\Gamma^J\delta A_J|\Psi_k\rangle\,.$$

This is just the usual formula from degenerate perturbation theory.

We would now like to compute the shift in the matrices, $\mathbf{X}_0^M \to \mathbf{X}_0^M + \delta\mathbf{X}^M$, generated by the shift in the gauge field. To do this, we first need to evaluate the quantity

$$\langle\Psi_{k''}|q_i|\delta\Psi_k\rangle = \sum_{\mathcal{E}_{k',n,s}\neq 0} \langle\Psi_{k''}|q_i|\Psi_{k',n,s}\rangle \frac{1}{2\mathcal{E}_{k',n,s}} \langle\Psi_{k',n,s}|\mathcal{D}\Gamma^J\delta A_J|\Psi_k\rangle\,. \tag{4.7}$$

Note that the q_i integral in $\langle\Psi_{k''}|q_i|\Psi_{k',n,s}\rangle$ will fix $k' = k''$. Moreover, since q_i has no effect on the spins, we may assume $s^i = \frac{1}{2}$. It is useful to rewrite the q^i in terms of raising and lowering operators,

$$q_i = -\frac{k_i}{f_i} + \frac{a_i + a_i^\dagger}{\sqrt{2f_i}}\,, \tag{4.8}$$

where the constant term, $-\frac{k_i}{f_i}$, arises because the center of our oscillator potential is shifted from the origin. Using the fact that $a_i|\Psi_k\rangle = 0$, and $\langle\Psi_k|\Psi_{k',n,s}\rangle = 0$ for $n \neq 0$, we see that

$$\langle\Psi_{k''}|q_i|\Psi_{k',n,s}\rangle = \langle\Psi_{k''}|\frac{a_i}{\sqrt{2f_i}}|\Psi_{k',n,s}\rangle\,, \tag{4.9}$$

which implies that we must have $|\Psi_{k',n,s}\rangle = a_i^\dagger|\Psi_{k'}\rangle$ for the matrix element not to vanish, and, hence, that

$$\langle\Psi_{k''}|q_i|\delta\Psi_k\rangle = \frac{1}{2f_i}\langle\Psi_{k''}|\frac{a_i}{\sqrt{2f_i}}\mathcal{D}\Gamma^J\delta A_J|\Psi_k\rangle\,.$$

Now, using equation (4.8), as well as the identities, $\langle\Psi_k|a_i^\dagger = 0$ and $\langle\Psi_k|\mathcal{D} = 0$ we get

$$\langle\Psi_{k''}|q_i|\delta\Psi_k\rangle = -\frac{1}{2f_i}\langle\Psi_{k''}|[\mathcal{D},q_i]\Gamma^J\delta A_J|\Psi_k\rangle = -\frac{1}{2f_i}\langle\Psi_{k''}|i\Gamma^{2i}\Gamma^J\delta A_J|\Psi_k\rangle\,. \tag{4.10}$$

This expression can be simplified still further:

$$-\frac{1}{2f_i}\langle\Psi_{k''}|i\Gamma^{2i}\Gamma^J\delta A_J|\Psi_k\rangle = -\frac{1}{f_i}\frac{i}{4}\langle\Psi_{k''}|(4i\Sigma^{2i,J}+2\delta_{2i,J})\delta A_J|\Psi_k\rangle\,, \tag{4.11}$$

where $\Sigma^{MN} = -\frac{i}{4}[\Gamma^M,\Gamma^N]$ is our Lorentz generator. Note that $\langle\Psi_{k''}|\Sigma^{2i,J}|\Psi_k\rangle$ vanishes unless $J = 2i-1$, in which case we can replace $\Sigma^{2i,2i-1}$ with $-\frac{1}{2}$. Hence we get

$$\langle\Psi_{k''}|q_i|\delta\Psi_k\rangle = -\frac{1}{2f_i}\langle\Psi_{k''}|\delta A_{2i-1}+i\delta A_{2i}|\Psi_k\rangle\,.$$

A similar calculation gives

$$\langle\Psi_{k''}|p_i|\delta\Psi_k\rangle = \frac{1}{2f_i}\langle\Psi_{k''}|\delta A_{2i}-i\delta A_{2i-1}|\Psi_k\rangle\,.$$

Consider that, under the perturbation by δA, the matrices $\mathbf{X}_0^J$ are shifted via $\mathbf{X}_0^J \to \mathbf{X}_0^J + \delta\mathbf{X}^J$, where

$$\delta\mathbf{X}^J_{k'k} = \langle\Psi_{k'}|x^J|\delta\Psi_k\rangle + \langle\delta\Psi_{k'}|x^J|\Psi_k\rangle\,.$$

Using the explicit form of $\langle\Psi_{k'}|x^J|\delta\Psi_k\rangle$, this reduces to simply

$$\delta\mathbf{X}^J_{k'k} = \langle\Psi_{k'}|\theta^{JK}\delta A_K|\Psi_k\rangle\,. \tag{4.12}$$

Note the similarity of this form of the shift in $\mathbf{X}^J$ to the standard expressions [10, 17, 18, 29, 30].

4.3 One final identity

Before we can derive the Born-Infeld action from the matrix action, we need one more identity. We need to understand the action, by commutator, of the $\mathbf{X}_0^J$ on a matrix of the form $\mathbf{G}_{k,k'} = \langle\Psi_k|g(x)|\Psi'_k\rangle$, where $g(x)$ is an arbitrary function. We have from equations (4.2) and (4.3) that

$$[\mathbf{X}_0^{2i},\mathbf{G}]_{k',k} = \frac{1}{f_i}(k_i-k'_i)\,(\mathbf{G})_{k',k}\ , \tag{4.13}$$

$$[\mathbf{X}_0^{2i-1},\mathbf{G}]_{k',k} = i(\partial_{k'_i}+\partial_{k_i})\,(\mathbf{G})_{k',k}\ . \tag{4.14}$$

Now consider,

$$\begin{aligned}\frac{1}{f_i}(k_i-k_i')\langle\Psi_{k'}|g(x^I)|\Psi_k\rangle &= \frac{1}{f_i}\int dx\ \left((k_i-k_i')\,\Psi^*_{k'}\Psi_k\, g(x^I)\right)\\ &= \frac{1}{f_i}\int dx\,(-i\partial_{p_i}(\Psi^*_{k'}\Psi_k))g(x^I)\\ &= \frac{1}{f_i}\int dx\,\Psi^*_{k'}\Psi_k(i\partial_{p_i}g(x^I))\\ &= \frac{1}{f_i}\langle\Psi_{k'}|(i\partial_{p_i}g(x^I))|\Psi_k\rangle\,. \end{aligned}\tag{4.15}$$

Similarly, we have

$$i(\partial_{k_i'}+\partial_{k_i})\langle\Psi_{k'}|g(x^I)|\Psi_k\rangle = \frac{1}{f_i}\langle\Psi_{k'}|(-i\partial_{q_i}g(x^I))|\Psi_k\rangle\,. \tag{4.16}$$

Taken together, these two identities tell us that we can replace the adjoint action of $\mathbf{X}_0^J$ with $i\theta^{JK}\partial_K$. Specifically, we have

$$[\mathbf{X}_0^J,\langle\Psi_k|f(x)|\Psi_k'\rangle] = \langle\Psi_k|i\theta^{JK}\partial_K f(x)|\Psi_k'\rangle\,. \tag{4.17}$$

Again, the form of this equation is expected, since the commutator is approximately given by the Poisson bracket at large field strength [18]. This gives us all the identities we need.

4.4 Derivation of the Born-Infeld action

Before we derive the full Born Infeld action, we consider first just the lowest order terms in $1/F^2$. These come from the lowest order term in the matrix action, given by

$$\frac{1}{4}\,\mathrm{tr}[\mathbf{X}_M,\mathbf{X}_N]^2\,,$$

whose equations of motion are

$$[\mathbf{X}^I,[\mathbf{X}_I,\mathbf{X}^J]] = 0\,.$$

Consider expanding the matrix equations of motion around $\mathbf{X}_0^I$ to first order in $\delta\mathbf{X}^I$,

$$\begin{aligned}\left[\mathbf{X}^I,[\mathbf{X}_I,\mathbf{X}^J]\right] &= \left[\mathbf{X}_0^I,[\mathbf{X}_{0I},\mathbf{X}_0^J]\right] + \left[\delta\mathbf{X}^I,[\mathbf{X}_{0I},\mathbf{X}_0^J]\right] + \\ &+ \left[\mathbf{X}_0^I,[\delta\mathbf{X}_I,\mathbf{X}_0^J]\right] + \left[\mathbf{X}_0^I,[\mathbf{X}_{0I},\delta\mathbf{X}^J]\right] + \mathcal{O}(\delta\mathbf{X}^2)\,. \end{aligned}\tag{4.18}$$

Using the fact that $[\mathbf{X}_0^I,\mathbf{X}_0^J]\propto\mathbf{1}$, we get the lowest order condition

$$[\mathbf{X}_0^I,[\mathbf{X}_{0I},\delta\mathbf{X}^J]] - [\mathbf{X}_0^I,[\mathbf{X}_0^J,\delta\mathbf{X}_I]] = 0\,.$$

Applying our formulas, (4.12) and (4.17), this equation reduces to

$$\langle\Psi_k|(-\theta^J{}_M\theta^{MK}\partial_J F_{KI})\theta^{MJ}|\Psi_{k'}\rangle = 0\,,$$

which can only vanish if

$$\theta^J{}_M\theta^{MK}\partial_J F_{KI} = 0\,.$$

As expected, this is nothing but the lowest order term in the $1/F$ expansion of the Born-Infeld equations of motion. This computation is similar to the one found in [18].

We now seek an all orders derivation. For this we require some knowledge of the terms in the matrix action to all powers of $\mathbf{X}^I$. The action is given by the dimensional reduction of the non-abelian Born Infeld action to $0+0$ dimensions. Unfortunately, the full non-abelian Born Infeld is not known beyond order α'^4 [40]. However, it is known that if one ignores terms with covariant derivatives and commutators of field strengths, then the full action is given by the symmetrized trace prescription found in [41].

Fortunately, it is easy to see that such terms cannot contribute in our approximation. Consider that, under dimensional reduction, we have $F_{MN} \to -i(2\pi\alpha')^{-2}[\mathbf{X}_N, \mathbf{X}_M]$. To zeroth order then, $F_{MN} \to (2\pi\alpha')^{-2}\theta_{MN}\mathbf{1}$, where we have used equation (4.4). Since we are expanding to lowest order, at most one F is not proportional to the identity. Hence, in a term with a commutator of two Fs, one of the two Fs must be proportional to the identity and the commutator must vanish.

A similar argument shows that terms with covariant derivatives cannot contribute. Under dimensional reduction, a covariant derivative is replaced by a commutator. Note that we must always have at least two derivatives and, using integration by parts, we can assume that the derivatives act on different Fs. Then, as we argued before, one of the two Fs must be proportional to the identity and will vanish when acted on by a commutator.

The upshot of this is that we may use as our all orders action [41]

$$\mathrm{STr}\,\sqrt{\det\left(\delta_{MN}\mathbf{1} - i(2\pi\alpha')^{-1}[\mathbf{X}_M, \mathbf{X}_N]\right)},$$

where STr stands for symmetrized trace and the determinant acts on the M, N indices. In this prescription one formally expands out the action in powers of the commutator and then sums with equal weight over each ordering of the commutators.

The equations of motion for this action are given by

$$\left[\mathbf{X}^I, \mathrm{S}\left\{\sqrt{\det\left(\delta_{MN}\mathbf{1} - i(2\pi\alpha')^{-1}[\mathbf{X}_M, \mathbf{X}_N]\right)}\left(\frac{C}{1-C^2}\right)_{IJ}\right\}\right] = 0\,, \tag{4.19}$$

where by S we mean symmetrization over all commutators and we have defined the matrix $C_{MN} = -i(2\pi\alpha')^{-1}[\mathbf{X}_M, \mathbf{X}_N]$. Note that this matrix is multiplied both through its spacetime indices and through ordinary matrix multiplication. Expanding equation (4.19) to first order in $\delta\mathbf{X}^K$ and using equations (4.12) and (4.17), we find

$$\sqrt{\det\left(\delta^{MN} + (2\pi\alpha')^{-1}\theta^{MN}\right)} \times \left(\frac{\tilde{\theta}^2}{1-\tilde{\theta}^2}\right)_{MN} \partial^M F^{NK} \left(\frac{\tilde{\theta}}{1-\tilde{\theta}^2}\right)_{KJ} = 0\,, \tag{4.20}$$

where $\tilde{\theta} = (2\pi\alpha')^{-1}\theta$. This is equivalent to the condition

$$\left(\frac{\tilde{\theta}^2}{1-\tilde{\theta}^2}\right)_{MN} \partial^M F^{NK} = \left(\frac{1}{1-(2\pi\alpha' F)^2}\right)_{MN} \partial^M F^{NK} = 0\,. \tag{4.21}$$

These are just the familiar equations of motion of the Born-Infeld action.

It is straightforward, although somewhat trivial, to extend these results to the full Dirac-Born-Infeld action by include transverse fluctuations of the brane. If X^A is a transverse direction, and we consider a shift δX^A, then the change in the matrices, to lowest order, is simply given by $\delta \mathbf{X}^A_{k'k} = \langle \Psi_{k'} | \delta X^A | \Psi_k \rangle$. In other words, to lowest order we can consider the zero modes unchanged. Substituting this shift in the matrices back into the matrix equations of motion gives the expected equations of motion for the transverse fluctuations.

It would be interesting to try to extend these results to the next order in δA_M. One could use the same procedure we have used here, but attempt to find the second order shift in the $\mathbf{X}^I$. This requires doing second order perturbation theory to find the new corrected zero modes. One could then attempt to use the matrix action to compute derivative corrections to the Born-Infeld action and compare with the results of [42].

5. Decoupling of the higher modes of the endpoint

In this section we would like to understand in detail how the higher modes of the endpoint decouple. To do this we consider the case of a flat D2-brane with uniform flux. This setup is convenient, as the associated world sheet theory is exactly solvable [38] and the dual D0-brane description takes a simple form. Moreover, as mentioned in the introduction, in this case there is a proof given by N. Ishibashi [10] that the two setups are equivalent.

As we will see below, the proof [10] of the equivalence of the D0-brane and D2-brane descriptions is essentially a proof that one may take the endpoint of the string to live in the ground state. Unfortunately, the proof given in [10] contains various singularities which we would like to avoid. Moreover, we would like to study the problem in a framework where we can unambiguously isolate the behavior of the endpoint of the string which cannot be done in the continuum framework considered in [10]. This motivates us to reconsider the argument of [10] in a latticized model of the string world sheet action.

In a latticized model, with lattice spacing ϵ, the endpoint at $\sigma = 0$ becomes a charged particle with mass ϵ that is coupled to its neighboring lattice site by a spring-like force with spring constant $1/\epsilon$. It is convenient to take the other endpoint at $\sigma = \pi$ to live on a brane without any magnetic flux, since otherwise the string spectrum would be independent of the flux [38].

If we ignore the coupling of the endpoint to its neighbor for the moment, the action of the endpoint is just that of the familiar Landau problem. Upon quantization, we find a ground state labeled by a momentum k and an infinite tower of evenly spaced excited states separated by an energy f/ϵ where f is the magnitude of the background magnetic field. Notice that as ϵ is taken to zero, the gap between the energy levels of the endpoints goes to infinity. This does not imply, however, that the endpoint is constrained to the ground state, as the force between the endpoint and its neighbor also goes to infinity in this limit.

To determine whether or not the endpoint leaves the ground state we perform the following computation, the details of which are given in appendix A: Construct the operators A and $A^\dagger$ which raise and lower the string from one energy level to another. The operator

$N = A^\dagger A$ is then the operator which gives the Landau level. We can then attempt to calculate the expectation value of N. As an example, consider $\langle 0|N|0\rangle$, where $|0\rangle$ is the ground state of the string, and take the limit as ϵ goes to zero. This calculation is not easy to perform analytically, but, in the large f limit it is given by

$$\langle 0|N|0\rangle = \frac{4}{3\pi}\,\frac{1}{f^3} + \mathcal{O}\left(\frac{1}{f^4}\right).$$

The important point here is that the average value of N is non-zero. We do see that, as f goes to infinity, the endpoint becomes constrained to the ground state as expected, but, away from this limit, there is a definite sense in which the endpoint is free to move to higher energy levels.

We would now like to study how these excitations of the endpoint decouple in the continuum limit. It is easy to check that, for any two finite norm states, ψ_1 and ψ_2, the inner product, $\langle\psi_1|A|\psi_2\rangle$, goes to zero as ϵ goes to zero. Indeed, the non-zero contribution of $\langle 0|N|0\rangle$ comes entirely from the normal ordering of $A^\dagger A$ in terms of the oscillators of the string. This implies that, in fact, for finite norm states ψ, the expectation value $\langle\psi|N|\psi\rangle$ is independent of the choice of ψ in the continuum limit.

What these results are saying is that physics below the cutoff scale is insensitive to whether or not the endpoint of the string is in the ground state. These results can be made even more definitive. As in [10], define a new hamiltonian, H_0, in which the endpoint is always constrained to its ground state. This hamiltonian is easily constructed by sending the mass of the endpoint to zero in the lagrangian and then canonically quantizing. It is the straightforward to show that

$$\lim_{\epsilon\to 0}(H - H_0) = 0\,.$$

In other words, the theory in which the string endpoint is constrained to the ground state and the theory in which it is allowed to leave the ground state are completely equivalent in the small ϵ limit. Note that this limit is only really valid when the expression is sandwiched between to finite norm fock space states. This decoupling of the endpoint excitations is the basic mechanism behind the duality between the D2-brane and the D0-brane descriptions. Indeed, the hamiltonian H_0 is precisely the hamiltonian of a string ending on an infinite collection of D0-branes with non-commutative coordinates $\mathbf{X}$ and $\mathbf{Y}$ satisfying $[\mathbf{X},\mathbf{Y}] = i/f$.

There is one aspect of this result which at first seems paradoxical. Consider that, in the full quantization of the string there is an oscillator b_0, which, for small f, raises the string from one Landau level to another. This seems to contradict the fact that the endpoint of the string can be assumed to be in the ground state.

To understand how this can happen, consider the following toy model similar to the one considered in [20]. Take a charged particle which is constrained to it's lowest energy level so that its coordinates obey $[x,y] = i/f$. Take a second particle of zero charge and mass m and couple it to the charged particle using a spring with spring constant k. The hamiltonian of this model is the same as the projected hamiltonian of the string H_0 in the case when there are only two lattice sites. The theory has two zero modes p and q which

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satisfy $[p, q] = i/f$, and two oscillators a and b. The hamiltonian is given by

$$H = \mathcal{E}_+ a^\dagger a + \mathcal{E}_- b^\dagger b\,,$$

where

$$\mathcal{E}_\pm = \sqrt{\frac{k}{m}}\sqrt{1 + \frac{mk}{2f^2}\left(1 \pm \sqrt{1 + \frac{4f^2}{mk}}\right)}.$$

Consider what happens when f is very small. In this case $\mathcal{E}_+$ becomes very large so that the a oscillator decouples. On the other hand $\mathcal{E}_-$ is given, to lowest order, by f/m. Thus, the hamiltonian is approximately given by $H = f/m b^\dagger b$. This hamiltonian is nothing but the ordinary Landau hamiltonian for a particle with mass m. In other words, when we couple a particle in the lowest Landau level to a massive particle, the massive particle behaves, in the weak field limit, as though it were a massive charged particle. Thus, the fact that the string can be raised to higher Landau levels is not inconsistent with the fact that the endpoint is trapped in the lowest Landau level.

6. Conclusions

We have now discussed some general features of smooth and fuzzy Dp-branes and constructed a map from one to the other. Moreover, we have checked in a few elementary examples that this map gives the expected results.

Unfortunately, we have had very little to say about smooth branes. The basic problem with understanding smooth branes is coming up with a consistent framework in which they can be studied. Indeed, at this stage we have only really defined them as branes which cannot be described as matrix valued configurations of D0-branes.

The example of the D2-brane with a single unit of flux should provide a natural laboratory for studying such branes, since the fuzzy description is completely inapplicable. In principle, one can try to manipulate the boundary state for such a brane in the same way as we did for the fuzzy case, but use a different regulator in which the $\epsilon \to 0$ limit, and the limit as the cutoff is removed are simultaneous. One can then try to expand the resulting boundary state around the small size limit to try and understand the physics of a single D0-brane expanding into a small D2-brane.

We have attempted this computation for a few choices of such a regulator and the results imply that one finds something like the boundary state of a D0-brane plus extra terms which are suppressed by factors of e^{-F} where F is the field strength on the D2-brane. Note that this non-perturbative suppression in the large F limit is required, as there is no mode living on a single D0-brane which allows it to expand into a D2-brane in perturbation theory. These results are suggestive, but one also finds various new divergences which need to be subtracted, and more information is required to extract any sensible physics from the computation.

Another subject which we have not discussed is whether other branes besides the flat brane with uniform flux have an exact duality between smooth and fuzzy descriptions. Here it seems likely that the BPS nature of the flat brane is important. It is not clear, however, how this works out in more general examples.

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A. Details of the lattice model

In this appendix, we give a few details of the lattice model discussed in section 5. We consider our brane to have only two coordinates X^1 and X^2. We take the string to have $n+1$ lattice sites with fields labeled X_m, where i runs from 0 to n. We take X_0 to be the endpoint of a string living on a brane with magnetic flux. The other endpoint X_n is taken to have Neumann boundary conditions. The lattice spacing, ϵ is given by π/n.

We take as our lattice model

$$\mathcal{L} = \sum_{m=0}^{n} \frac{1}{2}\epsilon \dot{X}_m^2 - \sum_{m=1}^{n} \frac{1}{2\epsilon}(X_m - X_{m-1})^2 - \frac{1}{2} f(X_0^1 \dot{X}_0^2 - X_0^2 \dot{X}_0^1)\,. \tag{A.1}$$

The model is solved in essentially the same way as the continuum model [38]. First we define the complex coordinates $X^{\pm} = (X^1 \pm iX^2)/\sqrt{2}$. We can then expand in terms of modes that solve the classical equations of motion,

$$X_m^+ = x^+ + i\sum_{k>0} a_k \psi_k(m) - i \sum_{k<0} b_{-k}^\dagger \psi_k(m)\,,$$

where

$$\psi_k^+(m) = \mathcal{N} \cos\left(\pi k \frac{m}{n} - \pi k\left(1 + \frac{1}{2n}\right)\right) e^{i\mathcal{E}_k \tau}\,.$$

The energies, $\mathcal{E}_k$, are given by the dispersion relation

$$\mathcal{E}_k = \frac{2}{\epsilon}\sin\left(\frac{\epsilon k}{2}\right),$$

and the momenta, k, are required to satisfy

$$f = -\frac{\sin(\pi k(1+1/n))}{\cos(\pi k(1+1/2n))}\,, \tag{A.2}$$

for $-n \le k \le n$. The normalization, $\mathcal{N}$, is determined by requiring $[a_k, a_k^\dagger] = 1$ and $[b_k, b_k^\dagger] = 1$. The full expression is quite complex, however, at large n, it is approximately given by $\mathcal{N} = \sqrt{|\mathcal{E}_k/\pi|}$.

We can now study the physics of the endpoint X_0. If we decouple the rest of the string, the endpoint lagrangian is just the usual lagrangian for a charged particle with mass ϵ in the presence of a magnetic field. Diagonalizing the endpoint hamiltonian just gives the usual Landau levels.

The raising and lowering operators that take us from one Landau level to another are given by $A = \epsilon\dot{X}_0^-/\sqrt{f}$ and $A^\dagger = \epsilon\dot{X}_0^+/\sqrt{f}$, which satisfy $[A, A^\dagger] = 1$. Note that since $\dot{X}_0$ is finite as $\epsilon \to 0$, both A and $A^\dagger$ go to zero as a formal expansion of oscillators.

On the other hand, consider defining the number operator $N = A^\dagger A$. If we denote normal ordering by ::, we can write $N =: A^\dagger A : +c_0$. where c_0 is a constant. The operator $: A^\dagger A :$ now simply goes to zero as $\epsilon \to 0$ when it is sandwiched between any two finite norm fock space states, but the constant c_0 does not. We have been unable to achieve an analytic expression for c_0, although a numerical estimate is quite easy to compute. However, if we assume f to be large, it is straightforward to expand the relation A.2 for large f and sum the series analytically. This gives

$$c_0 = \frac{4}{3\pi f^3} + \mathcal{O}\left(\frac{1}{f^4}\right),$$

as stated in section 5. In this sense, the one finds that the endpoint does leave the ground state.

Now we reproduce the argument of [10] by showing that in the $\epsilon \to 0$ limit the endpoint can be assumed to be in the ground state. The easiest way to put the endpoint into the ground state is just to remove its kinetic term. The only affect this has on our construction is to change the relation A.2 to

$$f = -\frac{\sin(\pi k)}{\cos(\pi k(1 + 1/2n))}. \tag{A.3}$$

At finite n, this has a definite effect on the energy spectrum of the theory. As $n \to \infty$, however, both relations (A.2) and (A.3) become simply $f = -\tan \pi k$, so that there is no effect on the spectrum. As a final note, a reader familiar with [10], may wonder what happened to the counter terms that were introduced in that version of the argument. In the latticized model they appear in the expansion of coupling of the endpoint to its neighboring lattice site and are, thus, naturally included.

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